ISSN 2229-5518

SEMILINEAR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH NONLOCAL CONDITIONS

M.Punitha, Anna University, Coimbatore, India. E mail: punithaarun77@yahoo.co.in. Dr.A.Santha, Sri Shakthi Institute of Engineering and Technology, Coimbatore, India.

ABSTRACT
The aim of this paper is to prove the existence and uniqueness of mild solutions for a semilinear differential equation of fractional order with nonlocal conditions in a Banach space. We also give an application for integro-partial differential equations of fractional order.
Keywords: *Nonlocal Cauchy problem, Functional differential equations, Fractional order, Mild solutions, Semigroups, Banach contraction theorem.*

1. INTRODUCTION

Fix $t_0 \geq 0$, a > 0, and $0 < a \leq 1$. Let $(E, \|\cdot\|)$ be a Banach space, F_i (i = 1,2), G, and f be continuous functions satisfying certain Lipschitz type inequalities, and let u_0 be an element of the Banach space E. In addition, let σ_i (i = 1,…,m) be certain continuous functions from interval $[t_0, t_0 + a]$ itself. The purpose of this paper is to study the existence and uniqueness of mild solutions of the following semilinear differential equations of fractional order with nonlocal conditions

$$\frac{d^{\alpha}u(t)}{dt^{\alpha}} + Au(t) = F_1(t,u(t),u(\sigma_1(t)),.....,u(\sigma_m(t)))$$
$$+\int_{t_0}^{t} F_2(t,u(t), \int_{to}^{s} f(t,s,u(s),\int_{to}^{s} h(s,\tau,u(\tau))d\tau)ds \quad ...(1.1)$$

$u(t_0) + G(u) = u_0$,

where $t \in (t_0, t_0 + a]$. In recent years several papers have been devoted to studying the existence of solutions of differential equations with nonlocal conditions (see e.g. (Kolodziej 2000; Byszewski 1991; Byszewski 1993; Dang 1993; Byszewski 1995; Chandrasekaran 2007; Xue 2005; Byszewski 1997; Henriquez *et al.* 2007)).

The nonlocal condition, which is a generalized of classical condition, was motivated by physical problem. For example in (Dang 1993).

2 EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS

Let E be a Banach space with norm $\|.\|$ and let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of operators on E.

In this paper we assume that - A is the infinitesimal generator of a C_0-semigroup of operators on E, D(A) is the domain of A, ≥ 0, a]> 0, I: = $[t_0,t_0 + a]$], $\{(t,s) : t_0 \leq s \leq t \leq t_0 + a]\} = \Delta$

In this section we establish the existence of mild solution to the system (1.1). In the sequel we write

$$M := \sup_{t\in[0,a]} \| T(t) \|_{BL(E,E)}$$

where BL (E,E) is the space of bounded linear operator from E to E, and we always assume that

$F_1 : I \times E^{m+1} \to E$, $F_2 : \Delta \times E^2 \to E$, $G : C(I, E) \to E$, $f: \Delta \times E \to E$, $h: \Delta \times E \to E$

$\sigma_i : I \to I$ (i = 1,….., m) are continuous functions. In the sequel, the operator norm $\|.\|_{BL(E,E)}$ will be donated by $\|.\|$.

Definition 2.1: A function $u \in C(I, E)$ satisfying the integral equation

$$u(t) = \int_0^{\infty} \xi_{\alpha}(\theta)T((t-t_0)^{\alpha}\theta)u_0 d\theta$$
$$-\int_0^{\infty} \xi_{\alpha}(\theta)T((t-t_0)^{\alpha}\theta)G(u)d\theta$$
$$+\alpha\int_0^{\infty}\int_{t_0}^{t} \theta(t-s)^{\alpha-1}\xi_{\alpha}(\theta)T((t-s)^{\alpha}\theta)$$
$$F_1(s,u(s),u(\sigma_1(s)),...,(\sigma_m(s)))d\theta ds +$$
$$\alpha\int_0^{\infty}\int_{t_0}^{t} \theta(t-s)^{\alpha-1}\xi_{\alpha}(\theta)T((t-s)^{\alpha}\theta)$$
$$(\int_{t_0}^{s} F_2(t,u(t),\int_{t_0}^{s} f(s,\tau,u(\tau),\int_{t_0}^{\tau} h(\tau,\mu,u(\mu))d\mu)d\tau)$$

$d\theta ds, t \in I.$ …(2.2)

is said to be a mild solution of nonlocal Cauchy problem on I where $\xi_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$,

$\int_0^\infty \xi_\alpha(\theta)d\theta = 1$ (see (E1-Borai 2002; E1-Borai *et at.* 2006; Mainardi 1995))

.

Definition 2.2: A function u:I → E is said to be a classical solution of the nonlocal Cauchy problem (1.1) on I if:

(i) u is continuous on I and continuously differential on $I \setminus \{t_0\}$,

(ii) $$\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = F_1(t,u(t),u(\sigma_1(t)),\dots,u(\sigma_m(t)))$$
$$+\int_{t_0}^t F_2(t,u(t), + \int_{t_0}^s f(t,s,u(s),\int_{t_0}^s h(s,\tau,u(\tau)d\tau ds, t \in I\backslash\{t_0\},$$ ... (1.1)

(iii) $u(t_0) + G(u) = u_0$,

Theorem 2.1: Assume that

(i) for all $Z_i \in E$ (i = 0, 1,….,m), the function $I \ni t \to F_1(t, Z_0, Z_1,\dots Z_m) \in E$ is continuous on I, for all $Z_i \in E$ (i =1,2) the function $\Delta \ni (t,s) \to F_2(t, s, Z_1, Z_2) \in E$ is continuous on Δ, for all $z \in E$ the function $\Delta \ni (t,s) \to f(t, s, z) \in E$ is continuous on Δ and $(t,s) \to f(t, s, z) \in E$, $(t,s) \to h(t, s, z) \in E$ $G:X \to E$, $\sigma_i \in C(I,I)$ (i=1, …,m) and $u_0 \in E$;

(ii) there are constant $L_i > 0$ (i=1,2,3,4) such that

$$\left\|F_1(t,z_0,z_1,\dots z_m) - F_1(t,\overline{z_0},\overline{z_1},\dots\overline{z_m})\right\| \le L_1\sum_{i=0}^m \|z_i - \overline{z_i}\|$$

for $t \in I$, $Z_i, \overline{Z}_i \in E(i = 0,1,\dots,m)$, ….. (2.3)

$$\left\|F_2(t,s,z_1,z_2) - F_2(t,s,\overline{z_1},\overline{z_2})\right\| \le L_2\sum_{i=1}^2 \left\|z_i - \overline{z_i}\right\|$$

for $(t,s) \in \Delta, Z_i, \overline{Z}_i \in E(i = 1,2)$, ….. (2.4)

$$\left\|f(t,s,z) - f(t,s,\overline{z})\right\| \le \frac{L3}{2}\left\|z - \overline{z}\right\|$$ for $(t,s) \in \Delta, Z, \overline{Z} \in E$,

….. (2.5)

$$\left\|h(t,s,z) - h(t,s,\overline{z})\right\| \le \frac{L3}{2}\left\|z - \overline{z}\right\|$$

for $(t,s) \in \Delta, Z, \overline{Z} \in E$, ….. (2.6)

$||G(w) - G(\overline{w})|| \le L_4 \quad \| w - \overline{w} \|_x$ ………(2.7)

(iii) $M[L_1 a^\alpha(m+1) + L_2 a^{\alpha+1}(1+L_3 a)+L_4] < 1$. ….. (2.8)

Then the nonlocal Cauchy problem (1.1) has a unique mild solution on I.

Proof: For $w \in C(I, E)$, we define the function $Qw: I \to E$ by

$$(Qw)(t) = \int_0^\infty \xi_\alpha(\theta)T((t-t_0)^\alpha\theta)u_0 d\theta$$
$$-\int_0^\infty \xi_\alpha(\theta)T((t-t_0)^\alpha\theta)G(w)d\theta\alpha\int_0^\infty\int_{t_0}^t \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)$$
$$F_1(s,w(s),w(\sigma_1(s)),\dots,w(\sigma_m(s)))d\theta ds +$$
$$\alpha\int_0^\infty\int_{t_0}^t \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\int_{t_0}^t F_2$$
$$(t,u(t),\int_{t_0}^s f(s,\tau,w(\tau),\int_{t_0}^\tau h(\tau,\mu,w(\mu))d\mu)d\tau)d\theta ds$$

It is easy to see that $Q(w) \in C(I, E)$. Moreover, Q is a contraction on C(I, E). In fact, for $w, \overline{w} \in C(I, E)$ we see that

$$(Qw)(t) - (Q\overline{w})(t) = -\int_0^\infty \xi_\alpha(\theta)T((t-t_0)^\alpha[G(w) - G(\overline{w})]d\theta$$
$$\alpha\int_0^\infty\int_{t_0}^t \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)$$
$$F_1(s,w(s),w(\sigma_1(s)),\dots,w(\sigma_m(s))) -$$
$$F_1(s,\overline{w}(s),\overline{w}(\sigma_1(s)),\dots,\overline{w}(\sigma_m(s)))]d\theta ds +$$
$$\int_{t_0}^t F_2(t,u(t), \int_{t_0}^s f(s,\tau,w(\tau), \int_{t_0}^\tau h(\tau,\mu,w(\mu)du)$$
$$-\int_{t_0}^t (F_2(t,u(t), \int_{t_0}^s f(s,\tau,\overline{w}(\tau),\int_{t_0}^t h(\tau,\mu,\overline{w}(\mu)d\mu)d\tau)d\theta ds.$$

and hence

$$\| (Qw)(t) - (Q\overline{w})(t) \| \le M\|G(w) - G(\overline{w})\| +$$
$$\alpha M\int_{t_0}^t (t-s)^{\alpha-1} \| F_1(s,w(s),w(\sigma_1(s)),\dots,w(\sigma_m(s))) -$$
$$F_1(s,\overline{w}(s),\overline{w}(\sigma_1(s)),\dots,\overline{w}(\sigma_m(s)))\| ds + \alpha M$$
$$\int_{t_0}^t (t-s)^{\alpha-1}\int_{t_0}^t F_2(t,w(t), \int_{t_0}^s f(s,\tau,w(\tau),$$
$$\int_{t_0}^s h(\tau,\mu,w(\mu)du) - \int_{t_0}^t (F_2(t,w(t), \int_{t_0}^s f(s,\tau,\overline{w}(\tau),$$
$$\int_{t_0}^t h(\tau,\mu,\overline{w}(\mu)d\mu)d\tau) \| ds.$$
$$\le M L_4 \| w - \overline{w}\|_x + \alpha M L_1 \int_{t_0}^t (t-s)^{\alpha-1} \| w(s) - \overline{w(s)} \|_x$$
$$+\sum_{i=1}^m \| w(\sigma_i(s) - \overline{w}\sigma_i(s)\|_x ds$$
$$+\alpha M L_2 \int_{t_0}^t (t-s)^{\alpha-1} \left(\int_{t_0}^s \|w(t) - \overline{w(t)}\| + \right.$$
$$\| \int_{t_0}^s f(s,\tau,w(\tau) - \int_{t_0}^s f(s,\tau,\overline{w}(\tau) \| +$$
$$\| \int_{t_0}^\tau h(\tau,\mu,w(\mu)d\mu) - \int_{t_0}^\tau h(\tau,\mu,w(\mu) \| d\mu)d\tau)ds$$

ISSN 2229-5518

$$\leq M L_4 \| w-\overline{w}\|_x + ML_1\alpha^{\alpha}(m+1) \ \| w-\overline{w}\|_x +$$

$$\alpha M L_2 \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{t_0}^{s} (\| w(\tau) - \overline{w(\tau)}\| +$$

$$L_3 \int_{t_0}^{t} (t-s)^{\alpha-1} \ \left(\int_{t_0}^{s} \| w(\mu) - \overline{w(\mu)}\| \, d\mu) d\tau\right) ds$$

$$= M[L_1 a^{\alpha}(m+1) + L_2 a^{\alpha+1}(1+L_3 a)) + L_4] \| w-\overline{w}\|_x,$$

which implies that Q is a contraction provided a > 0 is chosen in such a way that (2.7) is satisfied. Consequently, the operator Q satisfies the assumptions of the Banach contraction theorem. Therefore, in the space C(I, E) there is only one fixed point of the operator Q and this point is the mild solution of the nonlocal Cauchy problem (1.1). So the proof is complete.

3 APPLICATION

In order to apply the previous result we choose a special Banach space *E* and we specialize the function *G*. More precisely we consider the following nonlocal Cauchy problem

$$\frac{\partial^{\alpha} u(t,x)}{\partial t^{\alpha}} + \Delta u(t,x) = f_1(t,u(t),u(\sigma_1(t)),\dots\dots,u(\sigma_m(t))$$

$$+ f_2(t,u(t), \ + \ \int_{t_0}^{s} f(t,s,u(s),\int_{t_0}^{s} h(s,t,u(\tau)d\tau ds) \quad \dots(3.1)$$

$u(t,0) = u(t, \pi) = 0, \ t \geq 0$....... (3.2)

$$u_0(x) = u(t_0,x) + \sum_{i=1}^{\infty} c_i u(t_i,x), \ x \in [0,\pi] \quad \dots(3.3)$$

In the sequel,E=L² $[0,\pi]$ and A:D(A)contain E→ E is the operator Ay=Δ y with domain

D(A)={y€E:Δy€E,y(0)=y(π)=0}

It is well-known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t>0}$ on E. It is assumed that for certain constants $K_i > 0$; $i = 1$; 2; 3; 4; the following conditions are satisfied

$$\| f_1(t,v_0,v_1,\dots v_m) - f_1(t,v_0,v_1,\dots v_m)\| \leq K_1 \sum_{i=0}^{m} \| v_i - \overline{v}_i \|$$

for $t \in I, v_i, \overline{v}_i \in E(i=0,1,\dots,m)$, ….. (3.4)

$$\| f_2(t,s,v_1,v_2) - f_1(t,s,\overline{v_1},\overline{v_2})\| \leq K_2 \sum_{i=1}^{2} \| v_i - \overline{v}_i \|$$

for $(t,s) \in \Delta, \ v_i, \overline{v}_i \in E(i=1,2)$, ….. (3.5)

$$| \, | \quad f(t,s,v) - f(t,s,\overline{v}) \| \leq \frac{L3}{2} \| v-\overline{v} \| \text{for}$$

$(t,s) \in \Delta, v, \overline{v} \in E$, ….. (3.6)

$$| \, | \ h(t,s,v) - h(t,s,\overline{v}) | | \leq \frac{L_3}{2} \| v-\overline{v} \| \text{for } (t,s) \in \Delta, v, \overline{v} \in E,$$

….. (3.7)

(iii) $\|G(s) - G(\overline{s})\| \leq K_4 \|s-\overline{s}\|_{C(I:E)}$

for $s, \overline{s} \in C(I:E)$ …(3.8)

where $G(u(x)) = \sum_{i=1}^{\infty} c_i u(t_i,x)$

Consequently, theorem (2.1) can be applied for the equations (3.1)-(3.3) under the conditions (3.4)-(3.8).

REFERENCES

[1] Byszewski L (1991). Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal cauchy problem. *J. Math. Anal. Appl. 162*(2), pp. 494–505.

[2] Byszewski L (1993). Uniqueness criterion for solution of abstract nonlocal cauchy problem. *J.of applied mathematics and stochastic analysis 6*, pp. 49–54.

[3] Byszewski L (1995). Existence and uniqueness of mild and classical solutions of semilinear functional differential evolution nonlocal cauchy problem. *selected prob. in Math., Cracow Univ. of Tech.* (Anniversary Issue 6), pp. 25–33.

[4] Byszewski L (1997). On weak solutions of functional differential abstract nonlocal problem, *Annales Polonici Math 65*, pp. 163–170.

[5] Chandrasekaran M (2007), non local Cauchy problem for quasilinear integro differential equation in Banach space", electronic journal of differential equations, pp: 1-6, 2007.